

JACOB'S LADDERS AND SOME UNBOUNDED DECOMPOSITION OF THE ζ -OSCILLATING SYSTEMS ON PRODUCTS OF OTHER ζ -OSCILLATING SYSTEMS AS AN ζ -ANALOGUE OF THE PRIME DECOMPOSITION

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ABSTRACT. In this paper we introduce new class of multiplicative interactions of the ζ -oscillating systems generated by a subset of power functions. The main result obtained expresses an analogue of prime decomposition (without the property of uniqueness).

Dedicated to recalling of Nicola Tesla's oscillators

1. INTRODUCTION

1.1. Let us remind that in our papers [1] – [8] we have introduced within the theory of the Riemann zeta-function the following notions: Jacob ladders (JL), ζ -oscillating systems (OS), factorization formula (FF), metamorphosis of the oscillating systems (M), \mathcal{Z}_{ζ, Q^2} -transformations (ZT), and interactions between oscillating systems (IOS).

The main result obtained in this direction (IOS) in the paper [8] is the following set of ζ -analogues of the elementary trigonometric identity $\cos^2 t + \sin^2 t = 1$:

$$(1.1) \quad \cos^2(\alpha_0^{2,2}) \prod_{r=1}^{k_2} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{2,2}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^{2,2}\right)} \right|^2 + \\ + \sin^2(\alpha_0^{1,1}) \prod_{r=1}^{k_1} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{1,1}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^{1,1}\right)} \right|^2 \sim 1, \quad L \rightarrow \infty,$$

(for the notations used above see [8]).

Remark 1. We may call the kind of interactions of the ζ -oscillating systems in (1.1) as the linear interaction between the systems (LIOS), comp. section 2.4 in [8].

Of course, in our paper [8], we have obtained also other formulae for interactions of ζ -oscillating systems that are also good characterized by the words *linear interactions*, as in Remark 1.

1.2. In this paper we obtain different kind of the formula w.r.t. (1.1). Namely, we obtain new formula that is essentially non-linear multiplicative in corresponding

ζ -oscillating systems:

$$\begin{aligned}
 (1.2) \quad & \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r\right)}{\zeta\left(\frac{1}{2} + i\beta_r\right)} \right|^2 \sim \\
 & \sim \frac{1}{\Delta + 1} \prod_{l=1}^n (\Delta_l + 1) \frac{1}{(\alpha_0 - L)^\Delta} \prod_{l=1}^n (\alpha_0^{\Delta_l, k_l} - L)^{\Delta_l} \times \\
 & \times \prod_{l=1}^n \prod_{r=1}^{k_l} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{\Delta_l, k_l}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^{k_l}\right)} \right|^2, \quad L \in \mathbb{N}, \quad L \rightarrow \infty,
 \end{aligned}$$

where

$$1 \leq k_l \leq k_0, \quad \Delta = \sum_{l=1}^n \Delta_l, \quad \Delta > \Delta_1 \geq \Delta_2 \geq \dots \geq \Delta_n,$$

$$\Delta, \Delta_l \in \mathbb{R}^+, \quad k_0, n \in \mathbb{N},$$

for every fixed Δ, k_0, n .

Remark 2. A new property of interactions between the elements of set of ζ -oscillating systems generated over subset of class of power functions is expressed by our formula (1.2). Namely, given the main (say) ζ -oscillating system

$$(1.3) \quad \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r\right)}{\zeta\left(\frac{1}{2} + i\beta_r\right)} \right|^2 \leftarrow \Delta$$

is now factorized itself (comp. the factorization formula (3.8) in [8]) by means of the choice of basic system of the ζ -oscillating systems

$$(1.4) \quad \prod_{r=1}^{k_l} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{\Delta_l, k_l}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^{k_l}\right)} \right|^2 \leftarrow \Delta_l, \quad l = 1, \dots, n.$$

Remark 3. Let us remind the prime decomposition

$$(1.5) \quad n = p_1 \cdots p_l, \quad n \in \mathbb{N}.$$

Our formula (1.2) represents, in this direction, decomposition of the main ζ -oscillating system (1.3) into weighted product of basic ζ -oscillating systems (1.4). That is the formula (1.2) represents a ζ -analogue of the prime decomposition (1.5), except its uniqueness.

Let us remind the diagram (see [8]) of corresponding notions we have introduced within the theory of the Riemann zeta-function. The last lot in this diagram is IOS = interaction between oscillating systems. Now, the mentioned lot itself may be split into two others as follows

$$\text{IOS} \rightarrow \begin{cases} \text{LIOS} \\ \text{NIOS}, \end{cases}$$

about LIOS see Remark 1 and NIOS means non-linear IOS (for example, (1.2)).

2. LEMMA

2.1. This paper is devoted mainly to study of some type of multiplicative interactions between the ζ -oscillating systems generated by the following subset of the class of power functions

$$(2.1) \quad \begin{aligned} f(t) &= f(t; L, \Delta) = (t - L)^\Delta, \\ t &\in [L, L + U], \quad l \in \mathbb{N}, \quad \Delta > 0, \\ U &\in (0, a], \quad a \in (0, 1). \end{aligned}$$

Remark 4. The assumption

$$U \in (0, a], \quad a < 1$$

in (2.1) was chosen because of the possibility of interpretation of our results in terms of deterministic signals (pulses) in the communication theory. In this connection, see also the notion of the \mathcal{Z}_{ζ, Q^2} -transformation (device) we have introduced in our paper [7].

Since

$$f(L; L, \Delta) = 0$$

then we use, instead of Definition 2 from [8], the following

Definition. The symbol

$$f(t) \in \tilde{C}_0[T, T + U]$$

stands for the following

$$(2.2) \quad \begin{aligned} f(t) &\in C[T, T + U] \quad \wedge \quad f(t) \geq 0 \quad \wedge \\ &\wedge \quad \exists t_0 \in [T, T + U] : f(t_0) > 0, \\ T &> T_0, \quad U \in (0, U_0], \quad U_0 = o\left(\frac{T}{\ln T}\right), \quad T \rightarrow \infty. \end{aligned}$$

Remark 5. Of course,

$$\tilde{C} \subset \tilde{C}_0,$$

and

$$f(t; L, \Delta) \in \tilde{C}_0[L, L + U], \quad L > T_0, \quad \Delta > 0.$$

Remark 6. The case

$$\Delta \in (-1, 0)$$

is excluded though

$$\int_L^{L+U} (t - L)^\Delta dt < +\infty,$$

because

$$(t - L)^\Delta \rightarrow +\infty \quad \text{as } t \rightarrow L^+,$$

(the case $\Delta = 0$ is the trivial one).

Remark 7. We see that (2.2) implies the following

$$\int_T^{T+U} f(t) dt > 0,$$

and this is sufficient for applicability of our algorithm to generate of factorization formula (comp. [8], (3.1) – (3.11)).

2.2. Since (see (2.1))

$$(2.3) \quad \frac{1}{U} \int_L^{L+U} (t-L)^\Delta dt = \frac{1}{\Delta+1} U^\Delta, \quad \Delta > 0,$$

then we have by just mentioned algorithm (see Remark 7) the following

Lemma 1. For the function (2.1) there are vector-valued functions

$$(\alpha_0, \alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k), \quad k = 1, \dots, k_0, \quad k_0 \in \mathbb{N},$$

(k_0 being arbitrary and fixed) such that the following factorization formula holds true

$$(2.4) \quad \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r\right)}{\zeta\left(\frac{1}{2} + i\beta_r\right)} \right|^2 \sim \frac{1}{\Delta+1} \left(\frac{U}{\alpha_0 - L} \right)^\Delta, \\ L \in \mathbb{N}, \quad L \rightarrow \infty, \quad U \in (0, a], \quad a < 1, \quad \Delta > 0,$$

where

$$(2.5) \quad \begin{aligned} \alpha_r &= \alpha_r(U, L, \Delta, k), \quad r = 0, 1, \dots, k, \\ \beta_r &= \beta_r(U, L, k), \quad r = 1, \dots, k, \\ L < \alpha_0 < L + U &\Rightarrow 0 < \alpha_0 - L < U. \end{aligned}$$

Remark 8. Of course, in the asymptotic formula (2.4) the symbol \sim stands for (comp. [8], (3.8))

$$= \left\{ 1 + \mathcal{O}\left(\frac{\ln \ln L}{\ln L}\right) \right\} (\dots).$$

3. ON UNBOUNDED DECOMPOSITION OF ζ -OSCILLATING SYSTEMS

3.1. Now, we have, by our Lemma 1, that

$$(3.1) \quad (\Delta_l + 1)(\alpha_0^{\Delta_l, k_l} - L) \prod_{r=1}^{k_l} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{\Delta_l, k_l}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^{k_l}\right)} \right|^2 \sim U^{\Delta_l}, \\ L \rightarrow \infty, \quad \Delta_l > 0, \quad 1 \leq k_l \leq k_0,$$

where (see (2.5))

$$\Delta \rightarrow \Delta_l, \quad l \rightarrow k_l, \Rightarrow \alpha_r \rightarrow \alpha_r^{\Delta_l, k_l}, \dots$$

Consequently, we obtain by Lemma 1 and (3.1) the following

Theorem 1. In the case

$$(3.2) \quad \begin{aligned} \Delta &= \sum_{l=1}^n \Delta_l, \quad \Delta > \Delta_1 \geq \dots \geq \Delta_n, \\ \Delta, \Delta_l &> 0, \quad n \in \mathbb{N}, \end{aligned}$$

there are vector-valued functions

$$\begin{aligned} &(\alpha_0, \alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k), \\ &(\alpha_0^{\Delta_l, k_l}, \alpha_1^{\Delta_l, k_l}, \dots, \alpha_k^{\Delta_l, k_l}, \beta_1^{k_l}, \dots, \beta_k^{k_l}), \\ &l = 1, \dots, n \end{aligned}$$

where

$$\begin{aligned}\alpha_r &= \alpha_r(U, L, \Delta, k), \quad r = 0, 1, \dots, k, \\ \beta_r &= \beta_r(U, L, k), \quad r = 1, \dots, k, \\ \alpha_r^{\Delta_l, k_l} &= \alpha_r^{\Delta_l, k_l}(U, L, \Delta_l, k_l), \quad r = 0, 1, \dots, k_l, \\ \beta_r^{k_l} &= \beta_r^{k_l}(U, L, k_l), \quad r = 1, \dots, k_l,\end{aligned}$$

such that

$$\begin{aligned}(3.3) \quad & \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r\right)}{\zeta\left(\frac{1}{2} + i\beta_r\right)} \right|^2 \sim \\ & \sim \frac{1}{\Delta + 1} \prod_{l=1}^n (\Delta_l + 1) \frac{1}{(\alpha_0 - L)^\Delta} \prod_{l=1}^n (\alpha_0^{\Delta_l, k_l} - L)^{\Delta_l} \times \\ & \times \prod_{l=1}^n \prod_{r=1}^{k_l} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{\Delta_l, k_l}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^{k_l}\right)} \right|^2, \\ & L \in \mathbb{N}, \quad L \rightarrow \infty,\end{aligned}$$

where

$$\begin{aligned}L < \alpha_0 < L + U &\Rightarrow 0 < \alpha_0 - L < U, \\ L < \alpha_0^{\Delta_l, k_l} < L + U &\Rightarrow 0 < \alpha_0^{\Delta_l, k_l} - L < U,\end{aligned}$$

for arbitrary but fixed k_0, n .

Remark 9. The assumption of arbitrary but fixed n has its reason in the following

$$\prod_{l=1}^n \{1 + o(1)\} = 1 + o(1), \quad L \rightarrow \infty$$

that must be obeyed.

3.2. First, we give the following

Remark 10. In our paper [8] we have introduced some classes of additive interactions (linear ones in the corresponding non-linear ζ -oscillating systems). In contrast with these, our formula (3.3) expresses some class of multiplicative interactions of the corresponding ζ -oscillating systems.

Secondly, the formula (3.3) contains following products:

(a) the main (say) ζ -oscillating system

$$(3.4) \quad \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r\right)}{\zeta\left(\frac{1}{2} + i\beta_r\right)} \right|^2 \leftarrow \Delta \in \mathbb{R}^+,$$

(b) the basic set (say) of the ζ -oscillating systems

$$(3.5) \quad \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{\Delta_l, k_l}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^{k_l}\right)} \right|^2 \leftarrow \Delta_l \in \mathbb{R}^+, \quad l = 1, \dots, n,$$

(c) the product

$$(3.6) \quad \frac{1}{\Delta + 1} \prod_{l=1}^n (\Delta_l + 1)$$

that contains the generating parameters (see (3.2)),

(d) the product

$$(3.7) \quad \frac{1}{(\alpha_0 - L)^\Delta} \prod_{l=1}^n (\alpha_0^{\Delta_l, k_l} - L)^{\Delta_l}$$

that contains the control functions

$$(3.8) \quad \alpha_0, \alpha_0^{\Delta_l, k_l}.$$

Remark 11. The notion of *control functions* (3.8) is based on the fact that the quotient of the ζ -oscillating system (3.4) with the product

$$\prod_{l=1}^n \prod_{r=1}^{k_l} \left| \frac{\zeta \left(\frac{1}{2} + i\alpha_r^{\Delta_l, k_l} \right)}{\zeta \left(\frac{1}{2} + i\beta_r^{k_l} \right)} \right|^2$$

is proportional to the factor (3.7); the product (3.6) is constant for fixed representation (3.2) that is for corresponding interaction.

Remark 12. Consequently, the formula (3.3) expresses a kind of superfactorization of the main ζ -oscillating system (3.4). Namely, this main system is factorized itself by means of the basic of others ζ -oscillating systems (3.5). Of course, it is true that all the oscillating systems playing role are generated by the class of used power functions with $\Delta, \Delta_l \in \mathbb{R}^+$.

3.3. Now, we give, for example, the following extremal case of the superfactorization (3.3).

Corollary. If

$$k = k_1 = k_2 = \dots = k_n = 1,$$

then

$$(3.9) \quad \begin{aligned} & \left| \frac{\zeta \left(\frac{1}{2} + i\bar{\alpha}_1 \right)}{\zeta \left(\frac{1}{2} + i\beta_1 \right)} \right|^2 \sim \\ & \sim \frac{1}{\Delta + 1} \prod_{l=1}^n (\Delta_l + 1) \frac{1}{(\bar{\alpha}_0 - L)^\Delta} \prod_{l=1}^n (\alpha_0^{\Delta_l, 1} - L)^{\Delta_l} \times \\ & \times \prod_{l=1}^n \left| \frac{\zeta \left(\frac{1}{2} + i\alpha_1^{\Delta_l, 1} \right)}{\zeta \left(\frac{1}{2} + i\beta_1^1 \right)} \right|^2, \quad L \rightarrow \infty, \end{aligned}$$

where

$$\begin{aligned} \bar{\alpha}_r &= \alpha_r(U, L, \Delta, 1), \quad r = 0, 1, \\ \beta_1 &= \beta_1^1 = \beta_1(U, L, 1). \end{aligned}$$

Remark 13. Other extremal formulae correspond to the cases:

$$\begin{aligned} k &= k_0, \quad k_1 = k_2 = \dots = k_n = 1, \\ k &= 1, \quad k_1 = k_2 = \dots = k_n = k_0, \\ k &= k_1 = \dots = k_n = k_0. \end{aligned}$$

4. SOME ADDITIVE INTERACTIONS

4.1. If

$$(4.1) \quad \begin{aligned} f(t) &= f(t; L, \Delta_1, \dots, \Delta_n) = \sum_{l=1}^n (t-L)^{\Delta_l}, \\ t &\in [L, L+U], \quad \Delta_l > 0, \end{aligned}$$

of course,

$$f(t) \in \tilde{C}_0[L, L+U]$$

(comp. Definition), then

$$\frac{1}{U} \int_L^{L+U} \sum_{l=1}^n (t-L)^{\Delta_l} dt = \sum_{l=1}^n \frac{1}{\Delta_l + 1} U^{\Delta_l},$$

and, consequently, we obtain by our algorithm the following

Lemma 2. For the function (4.1) there are vector-valued functions

$$(\tilde{\alpha}_0, \tilde{\alpha}_1, \dots, \tilde{\alpha}_k, \tilde{\beta}_1, \dots, \tilde{\beta}_k), \quad k = 1, \dots, k_0$$

such that the following factorization formula

$$(4.2) \quad \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\tilde{\alpha}_r\right)}{\zeta\left(\frac{1}{2} + i\tilde{\beta}_r\right)} \right|^2 \sim \frac{\sum_{l=1}^n \frac{1}{\Delta_l + 1} U^{\Delta_l}}{\sum_{l=1}^n (\tilde{\alpha}_0 - L)^{\Delta_l}}, \quad \Delta_l > 0, \quad L \rightarrow \infty$$

holds true, where

$$\begin{aligned} \tilde{\alpha}_r &= \alpha_r(U, L, \Delta_1, \dots, \Delta_n, k), \quad r = 0, 1, \dots, k, \\ \tilde{\beta}_r &= \beta_r(U, L, 1), \quad r = 1, \dots, k.. \end{aligned}$$

Next, we have (see (3.1)) the following formula

$$(4.3) \quad \begin{aligned} (\alpha_0^{\Delta_l, k_l} - L)^{\Delta_l} \prod_{r=1}^{k_l} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{\Delta_l, k_l}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^{k_l}\right)} \right|^2 &\sim \frac{1}{\Delta_l + 1} U^{\Delta_l}, \\ \Delta_l &> 0, \quad 1 \leq k_l \leq k_0. \end{aligned}$$

Consequently, we have (see (4.2), (4.3)) the following

Theorem 2.

$$(4.4) \quad \begin{aligned} \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\tilde{\alpha}_r\right)}{\zeta\left(\frac{1}{2} + i\tilde{\beta}_r\right)} \right|^2 &\sim \\ &\sim \frac{\sum_{l=1}^n (\alpha_0^{\Delta_l, k_l} - L)^{\Delta_l}}{\sum_{l=1}^n (\tilde{\alpha}_0 - L)^{\Delta_l}} \prod_{r=1}^{k_l} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{\Delta_l, k_l}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^{k_l}\right)} \right|^2, \quad L \rightarrow \infty, \\ \Delta_l &> 0, \quad l = 1, \dots, n. \end{aligned}$$

Remark 14. The additive interaction (comp. [8]) between ζ -oscillating systems

$$\prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\tilde{\alpha}_r\right)}{\zeta\left(\frac{1}{2} + i\tilde{\beta}_r\right)} \right|^2 \leftarrow \{\Delta_1, \dots, \Delta_n\},$$

is expressed by the formula (4.4), and the set of n others oscillating systems

$$\prod_{r=1}^{k_l} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{\Delta_l, k_l}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^{k_l}\right)} \right|^2 \leftarrow \Delta_l, \quad l = 1, \dots, n.$$

Of course, we also may write the set of n similar formulae to (4.4).

Remark 15. Let us notice explicitly that the assumption (see (4.4))

$$\Delta_l > 0, \quad l = 1, \dots, n$$

on the set

$$\{\Delta_1, \dots, \Delta_n\}$$

is the only one (in comparison with the Theorem 1).

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